NEW GOSPER SPACE FILLING CURVES

HIROSHI FUKUDA
School of Administration and Informatics,
University of Shizuoka,
52-1 Yada, Shizuoka-shi, Shizuoka,
422-8526 JAPAN

MICHIO SHIMIZU
Department of Liberal Art,
Nagano Prefecture College,
8-49-7 Miwa, Nagano-shi, Nagano,
380-8525 JAPAN

GISAKU NAKAMURA
Research Institute of Education,
Tokai University,
2-28-4 Tomigaya, Shibuya-ku, Tokyo,
151-8677 JAPAN

ABSTRACT
We investigate several properties of a beautiful space filling curve known as the Gosper curve, and try to construct curves having the same properties. We found 17 new Gosper curves through computer searches, and an infinite series of curves that are natural extensions of the original Gosper curve.

KEY WORDS: Gosper curve, Space filling curves, Fractal

1. INTRODUCTION

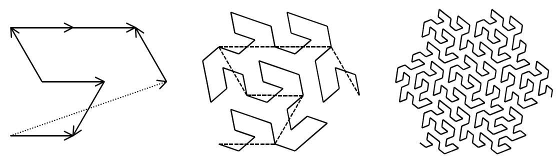
The Gosper curve is a space filling curve discovered by William Gosper, an American computer scientist, in 1973, and was introduced by Martin Gardner in 1976 [1,2]. The Gosper curve is said to be a very beautiful and complex monster curve because it has several special properties that other space filling curves such as the Peano curve and the dragon curve do not have.

In this paper, we point out several properties of the Gosper curve and try to construct curves having the same properties. Then, we show 17 new Gosper curves that we found through computer searches. We also show an infinite series of curves that are natural extensions of the original Gosper curve.

In section 2, we describe the Gosper curve. In section 3, we propose a construction method for new Gosper curves. In section 4, results are shown and discussed.

2. GOSPER SPACE FILLING CURVES

The Gosper curve is a recursive curve constructed by recursively replacing a dotted arrow, called the initiator, by seven arrows, called generator, as shown in Fig. 1(a). Fig. 1(b) and Fig. 1(c) illustrate the curves obtained by replacing the initiator by generator once and twice respectively.



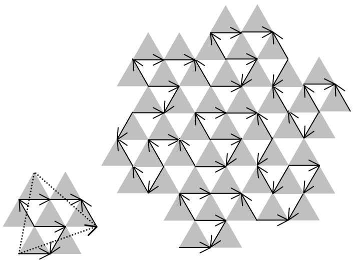
(a) (b) (c)

Fig. 1 Gosper curve.

The arrows of generator in the curve obtained after replacing the initiator to the generator any times are located on the edge of a regular triangular lattice. The degree of the generator arrow at the lattice point is two in the interior of the Gosper curve and is one at the both ends of initiator, i.e., the Gosper curve is the path from the root to the tip of the initiator without any branches visiting all interior lattice points. To our knowledge, the Gosper curve is the only space filling curve to have these properties. Therefore, in this paper we define the Gosper curve in terms of these properties. We then try to find new Gosper curves which satisfy our definition other than the Gosper curve found by William Gosper. We call the number of arrows included in the generator the size of generator or Gosper curve itself.

3. COMPUTER SEARCH FOR NEW GOSPER CURVES

We attach an equilateral triangle to the initiator arrow and to the generator arrow so that the triangle has the arrow as its edge and so that the direction of the arrow is counterclockwise in the triangle as shown in Fig. 2 (a). We call the equilateral triangle attached to each arrow a "flag".



(a) (b)

Fig. 2 Flags.

In the recursive procedure constructing Gosper curve, the flag for the initiator is replaced by the seven flags for the generator. The area of the flags is conserved during the replacement. Further, the flags fill the triangular lattice alternatively as shown in Fig. 2(b). From these simple flag properties, the following procedure to construct the generator of the Gosper curve can be deduced.

(P1) Chose the size of the Gosper curve N.

(P2) Prepare an initiator arrow of length L and attach the initial flag to it.

(P3) Prepare N flags with side length $L/\sqrt{N}$ to be used in the generator. The N flags should satisfy conditions (P4) to (P7). Bellow, we call the set of N flags "generator flags".

(P4) The generator flags should be connected at their vertices.

(P5) Two vertices of generator flags should coincide with the ends of the initiator. From this condition the size of Gosper curve is restricted to

$$N = x^2 + y^2 + xy, \qquad x, y = 0,1,2,... \qquad (1)$$

Therefore the possible sizes of Gosper curves are

$$N = 3,4,7,9,12,13,16,19,21,25,27,28,31,36,37,... \quad (2)$$

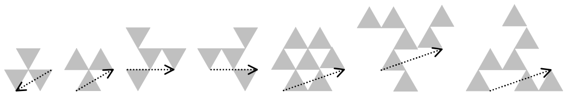
(P6) The shape of the generator flags is symmetric under a $2\pi/3$ rotation around its geometrical center.

(P7) If two coincident generator flags exist and one is translated by a distance L in a direction parallel to one of the three edges of a triangle in the initiator flag, then the two generator flags should not overlap.

(P8) By taking only one edge from each flag and by using the N edges of the generator flags, construct a path without branches from the root to the tip of the initiator.

(P9) Convert the edges in the path to the arrow so that the direction of the arrow is counterclockwise in the flag. We then obtain the generator arrow.

Using this procedure, we obtain the seven generator flags shown in Fig.3 for $N \le 7$. The generator flags which corresponds to the William Gosper's curve is the only generator flags satisfying conditions (P8) and (P9).



$N = 3$ $N = 4$ $N = 7$

Fig. 3 Generator flags for $N \le 7$.

4. RESULTS AND DISCUSSIONS

As we have already mentioned, the smallest Gosper curve obtained by the procedure given in the previous section is the curve with $N = 7$ found by William Gosper. For $7 < N \le 19$, there are two curves for $N = 13$ and $N = 19$. We show these curves together with their generators and generator flags in Fig. 4 and Fig. 5, respectively.

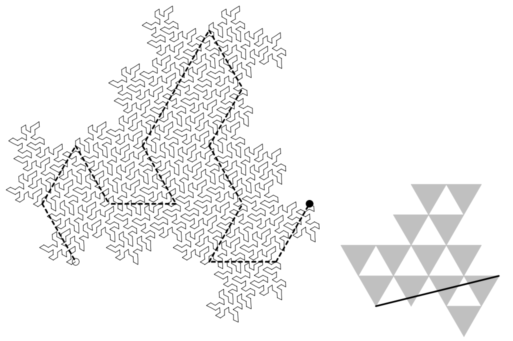


Fig. 4 A new Gosper curve for $N = 13$

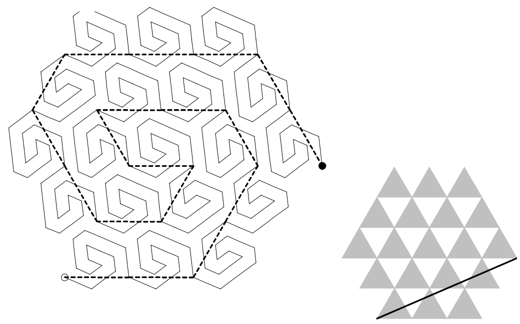
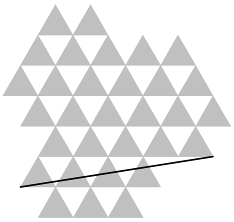
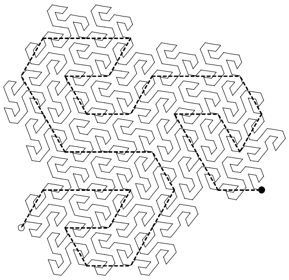


Fig. 5 A new Gosper curve for $N = 19$.

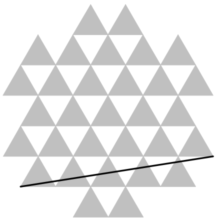
For $19 < N \le 37$, there are seven and eight curves for $N = 31$ and 37, respectively. The seven curves for $N = 31$ are constructed by the three generator flags shown in Fig.6 (31a), (31b) and (31c). Gosper curves which have the same size can be distinguished by using the flag and a serial number, for example (31a-1). It is notable that five new Gosper curves can be produced from the generator flags (31b). The curves produced from the same generator flags have similar shapes.



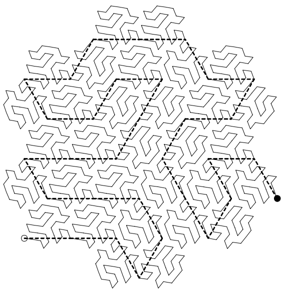
(31a)



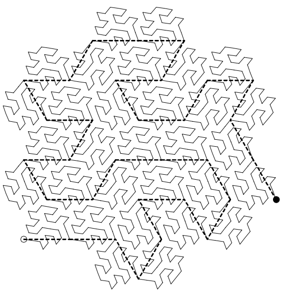
(31a-1)



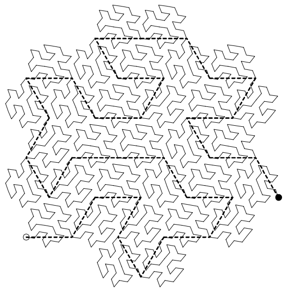
(31b)



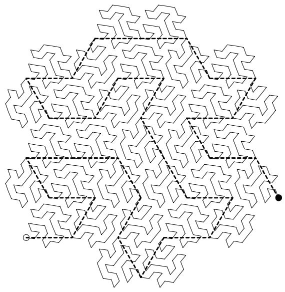
(31b-1)



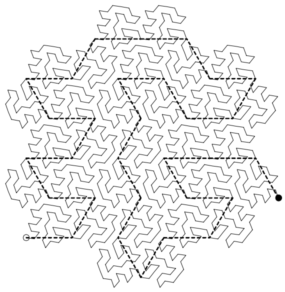
(31b-2)



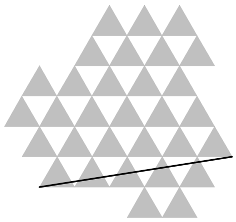
(31b-3)



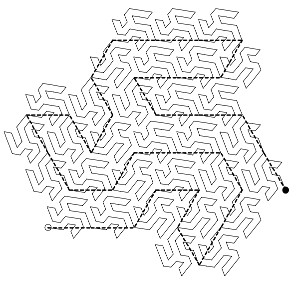
(31b-4)



(31b-5)



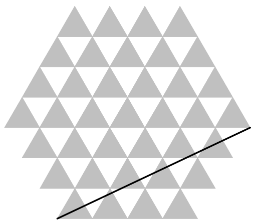
(31c)



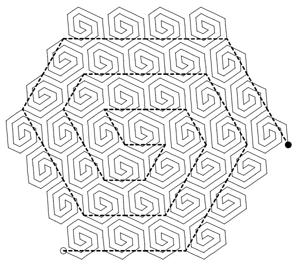
(31c-1)

Fig. 6 New Gosper curves for $N = 31$.

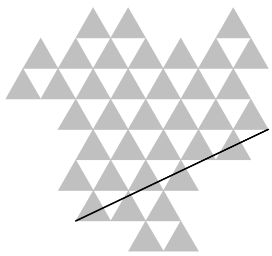
Eight curves for $N = 37$ are constructed from the four generator flags shown in Fig.7 (37a), (37b), (37c) and (37d). The generator flags (37a), (37b), (37c) and (37d) construct 1, 2, 3 and 2 curves, respectively. Although we did not search for curves with $N > 37$ because of the limitation of our computer power, we expect that there exist many big and complex Gosper curves.



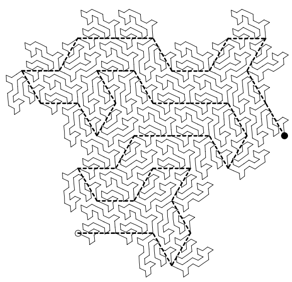
(37a)



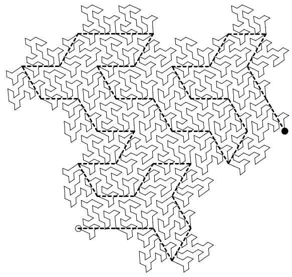
(37a-1)



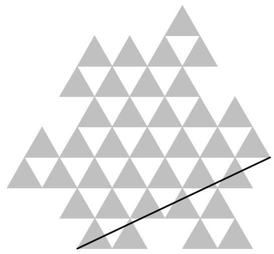
(37b)



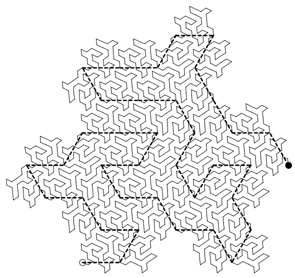
(37b-1)



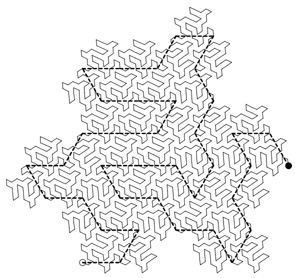
(37b-2)



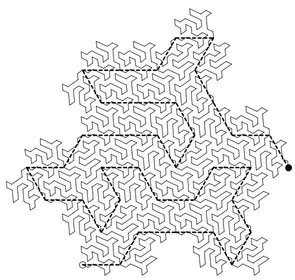
(37c)



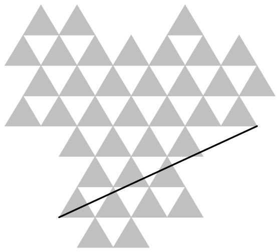
(37c-1)



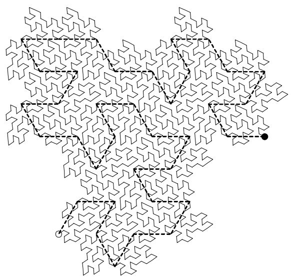
(37c-2)



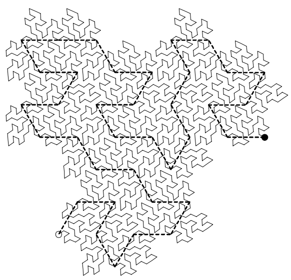
(37c-3)



(37d)



(37d-1)



(37d-2)

Fig. 7 New Gosper curves for $N = 37$.

Next, we compare the three curves with sizes $N = 7$, $N = 19$ and (37a-1). We found that the generator of the curve for $N = 19$ contains the generator of the curve for $N = 7$, and that the generator for (37a-1) contains the generator for $N = 19$. Similarly, the generator flags of the curve for $N = 19$ are obtained by adding a layer of flags around the generator flags of the curve for $N = 7$, and the generator flags for (37a-1) are obtained by adding a layer around the generator flags for $N = 19$. Thus the curve for $N = 19$ and the curve (37a-1) are natural extensions of the Gosper curve for $N = 7$.

It is possible to make the natural extension of the Gosper curve for $N = 7$ by adding a layer several times. We explain the extension method as an example of the curve for $N = 37$. First, we add a layer of flags around the generator flags for $N = 37$ as shown in Fig. 8. The resulting generator consists of 61 flags. Thus we will obtain the Gosper curve of the size $N = 61$. Next, we choose the ends of initiator to be the leftmost vertex on the button edge and the rightmost vertex, then we draw a dotted line from the ends of the initiator to the ends of the curve already drawn for $N = 37$ as shown in Fig. 8. After converting each segment of the dotted line to the arrow, we obtain the generator of Gosper curve for $N = 61$. It is clear that bigger curves can be obtained by continuing this procedure.

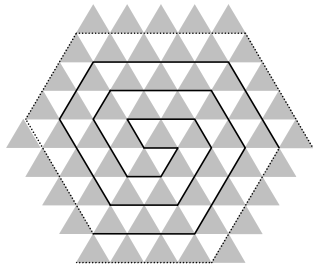


Fig. 8 Natural extension of the Gosper curve for $N = 61$

As mentioned above, by adding n layers to the generator flags for $N = 7$, we obtain a generator with size

$$N = 3n^2 + 3n + 1 \quad (n = 1,2,3,...). \tag{3}$$

Thus we can obtain an infinite series of Gosper curve with the size given by equation (3). The shape of the curves obtained looks like snail.

In this paper, we pointed out several properties of the monster curves found by William Gosper, and constructed new monster curves having the same properties. We found that the smallest size of the new monster curves is 13 and obtained an infinite series of natural extensions of the Gosper curve. These new recursive monster curves are beautiful and will be useful for computer graphics.

We should note that Gosper curve for $7 < N < 13$ may exist since the procedure (P1)—(P9) is not a necessary and sufficient condition for our definition of the Gosper curve. Finding a procedure corresponding to the necessary and sufficient conditions and applying the procedure to the other regular lattices are topics of future research.

REFERENCES

[1] M. Gardner, In which “monster” curves force redefinition of the word “curve”, *Scientific American* 235 (December issue), 1976, 124-133.

[2] B. B. Mandelbrot, *The fractal geometry*, (New York , W. H. Freeman and Company, 1977).