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Three-tangents theorem in three-body motion in three-dimensional space

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Abstract. In the general three-body problem on the plane, the conservation of the center of mass and zero angular momentum has a simple geometrical meaning: three tangent lines from the three bodies meet at a point at each instant. It is called "three-tangents theorem". Kuwabara and Tanikawa extended this theorem to the three-body motion in plane and in three-dimensional space with non-zero angular momentum. In this short note, we will investigate an alternative three-tangents theorem compared to Kuwabara and Tanikawa's in three-dimensional space.

1. Introduction

After the discovery of the figure-eight solution to the planar equal mass three-body problem [1,2,3,4], equal mass N-body periodic solutions has been paid match attention to. Recently new families of periodic solutions for three equal masses moving under Newtonian gravity in a plane were found by numerical simulation [5]. Both the figure-eight solution and new solutions have zero total linear momentum and zero total angular momentum. This fact leads to the following simple geometrical theorem [6]:

[**Theorem 1**] (*three-tangents*) *If the total linear momentum and the total angular momentum are zero, three tangent lines at bodies meet at a point or three lines are parallel.*

This theorem is proved without using the equation of motion of three bodies. It means that the three-tangents theorem can be applied to wide class of potential models. For example, it holds even for the three-body

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motion under the attractive logarithmic-potential accompanied by an artificial repulsive potential [6].

Kuwabara and Tanikawa [7] extended this theorem to the planar three-body motion with non-zero total angular momentum. They showed that

[**Theorem 2**] (*Extended three-tangents*) *If the total linear momentum is zero and the total angular momentum is* L *in the planar three-body motion, the area* $\boldsymbol{S}^t$ *of the triangle formed with three tangent lines at the bodies is given by*

$$\boldsymbol{S}^t \cdot \boldsymbol{\alpha} = \frac{L^2}{2},$$

where $\boldsymbol{\alpha}$ *is the double area of the triangle formed with three momentum vectors* $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$*:*

$$\boldsymbol{\alpha} = \mathbf{p}_1 \wedge \mathbf{p}_2 = \mathbf{p}_2 \wedge \mathbf{p}_3 = \mathbf{p}_3 \wedge \mathbf{p}_1.$$

The symbol $\wedge$ *is the exterior product of two vectors in two-dimensional space.*

Moreover, they applied the theorem to the three-dimensional case. When the total angular momentum is projected to the yz-, zx-, and xy-planes, and three tangent lines are also projected to each coordinate plane.

[**Theorem 3**] (*three-tangents in three dimensions I*) *If the total linear momentum is zero and the total angular momentum is* $\mathbf{L}$ *in the three-dimensional space, the area* $\boldsymbol{S}^t = (S^t_{yz}, S^t_{zx}, S^t_{xy})$ *of the triangle formed with projected three tangent lines to the* yz-*,* zx-*, and* xy-*planes is given by*

$$\boldsymbol{S}^t \cdot \boldsymbol{\alpha} = \frac{\mathbf{L} \cdot \mathbf{L}}{2},$$

where $\boldsymbol{L} = (L_{yz}, L_{zx}, L_{xy})$*, and* $\boldsymbol{\alpha} = (\alpha_{yz}, \alpha_{zx}, \alpha_{xy})$*.*

Being stimulated by Kuwabara-Tanikawa's study, we investigated alternative three-tangents theorem in three-dimensional space.

2. Alternative three-tangents theorem in three-dimensional space

We will set up a Cartesian coordinate system O-xyz. Suppose that three bodies with masses m_1, m_2, m_3 are acting the inertial forces between any two bodies obeying Newton's third law in the space. According to the equation of motions, three bodies are configured in the space keeping the center of mass being fixed in the Cartesian coordinate system. It is convenient to take the center of mass of three bodies as the origin O in the system O-xyz.

Also each body has its own momentum vector $\mathbf{p}_i$ $(i = 1, 2, 3)$ so that they satisfy $\sum_i \mathbf{p}_i = \mathbf{0}$. Three lines l_i $(i = 1, 2, 3)$ are drawn along $\mathbf{p}_i$ so that each line passes through each position for each body. If one of momentum vectors is zero vector, put a point in the space instead of drawing a line. We will call l_1, l_2, l_3 tangent lines. Let us introduce a vector $\boldsymbol{\alpha}$ which is the double area of the triangle formed with three momentum vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$:

$$\boldsymbol{\alpha}=\mathbf{p}_1 \times \mathbf{p}_2 = \mathbf{p}_2 \times \mathbf{p}_3 = \mathbf{p}_3 \times \mathbf{p}_1,$$

where the symbol $\times$ is the vector product of two three-dimensional vectors. This relation is directly derived from $\sum_i \mathbf{p}_i = \mathbf{0}$. Since three momentum vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are perpendicular to the double area $\boldsymbol{\alpha}$, they are always parallel to a plane if all three momentum vectors are not zero vectors.

If $\boldsymbol{\alpha}=\mathbf{0}$, either one of three momentum vectors is zero vector and others are parallel with each other, or three momentum vectors are parallel with one another, or all three momentum vectors are zero vectors.

On the other hand, If $\boldsymbol{\alpha}\neq \mathbf{0}$, three momentum vectors form a triangle in the three-dimensional space. In this case, the following theorem is valid:

[**Theorem 4**] (*three-tangents in three dimensions II*) *If the total linear momentum is zero and three tangent lines are projected to a plane including the total angular momentum in the three-dimensional space, the projected three tangent lines meet at a point or three tangent lines are parallel.*

3. Proof of Theorem 4

First, we will set up a Cartesian coordinate system O-$x'y'z'$. Three vectors $\hat{\mathbf{x}}' = (1,0,0)$, $\hat{\mathbf{y}}' = (0,1,0)$, and $\hat{\mathbf{z}}' = (0,0,1)$ are taken as unit vectors along x'-, y'-, and z'-axis, respectively. Let us choose a piece of plane τ passing through the origin O and being parallel to the total angular momentum $\mathbf{L}$. The normal vector of τ is denoted by $\boldsymbol{n}$. Now we set Cartesian coordinate system O-$x'y'z'$ so that two orthogonal x'- and y'-axis are on τ, the rest z'-axis is perpendicular to τ.

Second, we will project position and momentum vectors ($\mathbf{q}_i$ and $\mathbf{p}_i$) of three bodies onto τ. The projected position and momentum vectors are denoted by $\mathbf{q}'_i$ and $\mathbf{p}'_i$. To project position and momentum vectors in the three-dimensional space onto τ, it is convenient to introduce the projection matrix P in terms of $\boldsymbol{n}$ by $P = I - \boldsymbol{n}\,{}^t\boldsymbol{n}$, where I is the identity matrix. Then we obtain the relations between $\mathbf{q}_i$ and $\mathbf{q}'_i$, similarly between $\mathbf{p}_i$ and $\mathbf{p}'_i$:

$$\begin{aligned}\mathbf{q}'_i &= P\mathbf{q}_i \\ &= \mathbf{q}_i - (\mathbf{q}_i \cdot \boldsymbol{n})\boldsymbol{n},\end{aligned} \tag{1}$$

and

$$\begin{aligned}\mathbf{p}'_i &= P\mathbf{p}_i \\ &= \mathbf{p}_i - (\mathbf{p}_i \cdot \boldsymbol{n})\boldsymbol{n}.\end{aligned} \tag{2}$$

Note that $\mathbf{q}_i'$ and $\mathbf{p}_i'$ are lying on the plane τ at each instant. For the system O-$x'y'z'$ in which $\boldsymbol{n}$ is along z'-axis, the component representations are written by

$$\mathbf{q}_i' = (q_{ix'}, q_{iy'}, 0), \quad \mathbf{p}_i' = (p_{ix'}, p_{iy'}, 0).$$

Since the center of mass O of three bodies is projected onto itself, the conservation laws of the center of mass and the total linear momentum are valid for the Cartesian coordinate system O-$x'y'z'$. Thus we have $\sum_i \mathbf{q}_i' = \mathbf{0}$ and $\sum_i \mathbf{p}_i' = \mathbf{0}$ at every instant. The total angular momentum is evaluated in the system O-$x'y'z'$ as follows:

$$\begin{aligned}\mathbf{L} &= \sum_i \mathbf{q}_i' \times \mathbf{p}_i' + \sum_i (q_{iz'}p_{ix'} - q_{ix'}p_{iz'})\hat{\mathbf{y}}' + \sum_i (q_{iy'}p_{iz'} - q_{iz'}P_{iy'})\hat{\mathbf{x}}' \\ &= \mathbf{L}_{z'} + \mathbf{L}_{y'} + \mathbf{L}_{x'}.\end{aligned}$$

On the other hand the total angular momentum $\mathbf{L}$ is always on the plane τ, so only the z' component of $\mathbf{L}$ varnishes:

$$\begin{aligned}\mathbf{L}_z' &= \sum_i \mathbf{q}_i' \times \mathbf{p}_i' \\ &= \sum_i (q_{ix'}p_{iy'} - q_{iy'}p_{ix'})\hat{\mathbf{z}}' = \mathbf{0}.\end{aligned}$$

Now let $\mathbf{c}_t'$ be the intersection point of two projected tangent lines l_1' and l_2'. The angular momentum $\mathbf{L}_z'$ about the intersection point $\mathbf{c}_t'$ is also zero: $\sum_i (\mathbf{q}_i' - \mathbf{c}_t') \times \mathbf{p}_i' = \mathbf{0}$. Also $(\mathbf{q}_1' - \mathbf{c}_t') \times \mathbf{p}_1' = \mathbf{0}$ and $(\mathbf{q}_2' - \mathbf{c}_t') \times \mathbf{p}_2' = \mathbf{0}$. Thus we have $(\mathbf{q}_3' - \mathbf{c}_t') \times \mathbf{p}_3' = \mathbf{0}$. This means that the projected tangent three lines$l_1', l_2'$and l_3' meet at the same point $\mathbf{c}_t'$ on τ. □

Before closing our statement, we will represent the intersection point of three tangent lines $\mathbf{c}_t'$ on τ by the projected position and momentum vectors. It is simply written by

$$\boldsymbol{c}_t' = -\frac{[(\mathbf{q}_i' \times \mathbf{p}_i') \cdot \hat{\mathbf{z}}']\mathbf{p}_j' - [(\mathbf{q}_j' \times \mathbf{p}_j') \cdot \hat{\mathbf{z}}']\mathbf{p}_i'}{(\mathbf{p}_i' \times \mathbf{p}_j') \cdot \hat{\mathbf{z}}'}, \tag{3}$$

where $(i, j) = (1, 2), (2, 3), (3, 1)$. Substituting (1) and (2) into (3), $\boldsymbol{c}_t'$ can also be written by $\mathbf{q}_i$, $\mathbf{p}_i$ and normal vector $\boldsymbol{n}$ in the three-dimensional space.

Let us rotate τ in the three-dimensional space for fixed time. The intersection point $\boldsymbol{c}_t'$ goes to infinity if the plane τ is parallel to $\boldsymbol{\alpha}$ ($\boldsymbol{n}$ is perpendicular to $\boldsymbol{\alpha}$). This is easily verified because

$$(\mathbf{p}_i' \times \mathbf{p}_j') \cdot \hat{\mathbf{z}}' = (\mathbf{p}_i \times \mathbf{p}_j) \cdot \boldsymbol{n} = \boldsymbol{\alpha} \cdot \boldsymbol{n} = 0.$$

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