Coloring of Polytopes in Four Dimensions

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Abstract: We report on the enumeration of coloring of the regular polytopes in four dimensions by using Polya's theorem. Results are given by the cycle indices for the regular polytope of 5-, 8-, 16- and 24-cell.

1 Introduction

In this paper, we enumerate the coloring of the polytope in four dimensions (Coxeter 1973) using the Polya's theorem of combinatorial theory (Riordan 1978). The coloring of the polyhedron in three dimensions is frequently introduced as an example of the Polya's theorem. In higher dimensions, it is not so easy to enumerate the coloring of polytopes.

We investigate four kinds of colorings; the vertex-coloring, edge-coloring, face-coloring and cell-coloring. The vertex-coloring of the polyhedron (polytope in higher dimensions) is an assignment of colors to its vertices. Likewise, edge-, face-, cell-coloring are assignments of colors to the edges, faces, and cells respectively.

In section 2, we enumerate the coloring of the polyhedron in three dimensions using the Polya's theorem in order to explain the 'coloring' problem in detail and how to apply the theorem. In section 3, we describe the computational method to enumerate the coloring of polytopes in four dimensions by using computer. We also discuss the results given by the cycle indices for the regular polytope of 5-, 8-, 16- and 24-cell.

2 Coloring of a cube

In order to explain the 'coloring' problem and also the Polya's theorem, we describe the enumeration of the coloring of a cube in three dimensions. The theorem gives the enumerating generating function (or simply the enumerator) of ordered n

things chosen independently from a store having enumerator $S(x_1, x_2, \cdots)$ and taken as equivalent (non-distinct) if there is a permutation of a group G which sends one into the other. The enumerator $F_n(x_1, x_2, \cdots)$ is given by

$$F_n(x_1, x_2, \cdots) = H_n(S_1, S_2, \cdots, S_n). \tag{1}$$

In equation (1), $H_n(t_1, t_2, \cdots, t_n)$ is called the cycle index of a permutation of a group G, and it is defined by

$$H_n(t_1, t_2, \cdots, t_n) = \frac{1}{|G|}\sum_{\pi \in G} C(\pi),\ C(\pi) = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}. \tag{2}$$

where i_1 is the number of 1-cycles and i_2 of 2-cycles, and so on, in $\pi \in G$. S_k is the enumerator for choices of k objects from the store which remain invariant under cyclic permutation of length k (the only choices invariant for cycles of length k are those for which all objects are alike).

We enumerate the face-coloring of a cube using the six colors, the number of faces. In this paper, the number of colors does not necessarily indicate the number of colors actually being used, but it simply means that it is possible to use up to that number of colors. We do not distinguish two colorings if they can be identical by a suitable rotation.

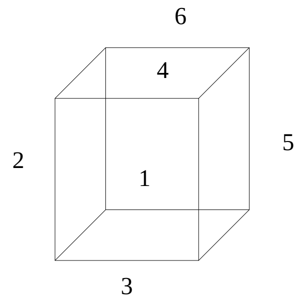

Fig. 1. Cube.

We label the six faces by 1 to 6 as shown in Fig. 1. The group G consists of the 24 permutations which represent the rotation of the cube. Every element π of the G is tabulated in Table 1 along with $C(\pi)$. The cycle index of G is thus given by

$$H_6(t_1, t_2, \cdots, t_6) = \frac{1}{24}(t_1^6 + 3t_1^2 t_2^2 + 6t_1^2 t_4^1 + 6t_2^3 + 8t_3^2). \tag{3}$$

Substitution of the enumerator S_k for six colors

$$S_k = x_1^k + x_2^k + \cdots + x_6^k \tag{4}$$

to equation (3) yields the enumerator $F_6(x_1, x_2, \cdots, x_6)$ for the face-coloring of cube by six colors. The term $x_1^{i_1} x_2^{i_2} \cdots x_6^{i_6}$ represents that i_1 faces are colored by the color x_1, and i_2 faces are colored by the color x_2, and so on. Therefore if we put $x_1 = x_2 = \cdots = x_6 = 1$ in the enumerator $F_6(x_1, x_2, \cdots, x_6)$, we obtain the number 2226 of the face-coloring of cube by six colors.

For vertex-coloring of cube, the cycle index

$$H_8(t_1, t_2, \cdots, t_8) = \frac{1}{24}(t_1^8 + 8t_1^2 t_3^2 + 9t_2^4 + 6t_4^2) \tag{5}$$

of its eight vertices and the enumerator S_k for eight colors

$$S_k = x_1^k + x_2^k + \cdots + x_8^k \tag{6}$$

gives the number 701968 of vertex-coloring of cube by eight colors. Finally, the cycle index of 12 edges

$$H_{12}(t_1, t_2, \cdots, t_{12}) = \frac{1}{24}(t_1^{12} + 6t_1^2 t_2^5 + 3t_2^6 + 6t_4^3 + 8t_3^4) \tag{7}$$

gives the number 371513523888 of edge-coloring of cube by 12 colors.

Table 1. Elements π and $C(\pi)$ of group G for rotation of a cube

π	$C(\pi)$	π	$C(\pi)$	π	$C(\pi)$
$\begin{pmatrix}123456\\123456\end{pmatrix}$	t_1^6	$\begin{pmatrix}123456\\624351\end{pmatrix}$	$t_1^2t_2^2$	$\begin{pmatrix}123456\\462513\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\135246\end{pmatrix}$	$t_1^2t_4^1$	$\begin{pmatrix}123456\\326154\end{pmatrix}$	$t_1^2t_4^1$	$\begin{pmatrix}123456\\536142\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\154326\end{pmatrix}$	$t_1^2t_2^2$	$\begin{pmatrix}123456\\456123\end{pmatrix}$	t_2^3	$\begin{pmatrix}123456\\365214\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\142536\end{pmatrix}$	$t_1^2t_4^1$	$\begin{pmatrix}123456\\351624\end{pmatrix}$	t_2^3	$\begin{pmatrix}123456\\541632\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\263415\end{pmatrix}$	$t_1^2t_4^1$	$\begin{pmatrix}123456\\632541\end{pmatrix}$	t_2^3	$\begin{pmatrix}123456\\312564\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\653421\end{pmatrix}$	$t_1^2t_2^2$	$\begin{pmatrix}123456\\645231\end{pmatrix}$	t_2^3	$\begin{pmatrix}123456\\231645\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\513462\end{pmatrix}$	$t_1^2t_4^1$	$\begin{pmatrix}123456\\564312\end{pmatrix}$	t_2^3	$\begin{pmatrix}123456\\246135\end{pmatrix}$	t_3^2
$\begin{pmatrix}123456\\421653\end{pmatrix}$	$t_1^2t_4^1$	$\begin{pmatrix}123456\\214365\end{pmatrix}$	t_2^3	$\begin{pmatrix}123456\\415263\end{pmatrix}$	t_3^2

3 Coloring of regular polytopes in four dimensions

In order to enumerate the coloring of polytopes in four dimensions, the rotation group G stated above is necessary. We obtain the element π of the G from the permutation P of vertices. The permutation P which satisfies the following conditions is the element of the rotation group G. (A) For all edges, set of vertices of an edge is transformed by permutation P into that of another edge or stays the same. (B) The same goes for vertices of all faces. (C) The same goes for vertices of all cells. (D) Determinant of the transformation matrix of coordinates of vertex which corresponds to P is positive. In order to complete the whole permutation, we initially set a part of the permutation which satisfies the conditions (A)—(C). When all three are met, we continue the permutation to its completion.

For the regular polytope of 5-cell, we obtain cycle indices

$$H_5(t_1,t_2,\cdots,t_5)=\frac{1}{60}(t_1^5+20t_1^2t_3^1+15t_1^1t_2^2+24t_5^1),$$

of its 5 vertices for vertex-coloring and

$$H_{10}(t_1,t_2,\cdots,t_{10})=\frac{1}{60}(t_1^{10}+20t_1^1t_3^3+15t_1^2t_2^4+24t_5^2).$$

of its 10 edges for edge-coloring. The cycle indices for face-coloring and cell-coloring are identical with those for edge-coloring and vertex-coloring, respectively, because of the self-duality.

For the regular polytope of 8-cell, we obtain cycle indices

$$H_{16}(t_1,t_2,\cdots,t_{16})=\frac{1}{192}(t_1^{16}+32t_1^4t_3^4+12t_1^4t_2^6+31t_2^8+36t_4^4+48t_8^2+32t_2^2t_6^2),$$

of its 16 vertices for vertex-coloring,

$$H_{32}(t_1,t_2,\cdots,t_{32})=\frac{1}{192}(t_1^{32}+32t_1^2t_3^{10}+19t_2^{16}+24t_1^4t_2^{14}+36t_4^8+48t_8^4+32t_2^1t_6^5),$$

of its 32 edges for edge-coloring,

$$\begin{aligned}H_{24}(t_1,t_2,\cdots,t_{24})=\frac{1}{192}(&t_1^{24}+t_2^{12}+32t_3^8+18t_1^4t_2^{10}\\&+24t_1^2t_2^{11}+12t_1^4t_4^5+48t_8^3+32t_6^4+12t_4^6+12t_2^2t_4^5),\end{aligned}$$

of its 24 faces for face-coloring, and

$$\begin{aligned}H_8(t_1,t_2,\cdots,t_8)=\frac{1}{192}(&t_1^8+32t_1^2t_3^2+13t_2^4+24t_1^2t_2^3+12t_1^4t_4^1\\&+48t_8^1+6t_1^4t_2^2+32t_2^1t_6^1+12t_4^2+12t_2^2t_4^1),\end{aligned}$$

of its 8 cells for cell-coloring. The cycle indices of the regular polytope of 16-cell for vertex-coloring, edge-coloring, face-coloring, and cell-coloring are identical with those of 8-cell for cell-coloring, face-coloring, edge-coloring, and vertex-coloring, respectively, because of duality between 16-cell and 8-cell.

For the regular polytope of 24-cell, we obtain cycle indices,

$$H_{24}(t_1,t_2,\cdots,t_{24}) = \frac{1}{576}(t_1^{24} + t_2^{12} + 32t_1^6t_3^6 + 18t_1^4t_2^{10} + 36t_1^4t_4^5 + 72t_1^2t_2^{11} + 144t_8^3 + 48t_3^8 + 96t_{12}^2 + 48t_6^4 + 32t_2^3t_6^3 + 12t_4^6 + 36t_2^2t_4^5),$$

of its 24 vertices for vertex-coloring and

$$H_{96}(t_1,t_2,\cdots,t_{96}) = \frac{1}{576}(t_1^{96} + 32t_1^6t_3^{30} + 19t_2^{48} + 84t_4^{24} + 72t_1^4t_2^{46} + 144t_8^{12} + 48t_3^{32} + 96t_{12}^8 + 48t_6^{16} + 32t_2^3t_6^{15}).$$

of its 96 edges for edge-coloring. The cycle indices for face-coloring and cell-coloring are identical with those for edge-coloring, and vertex-coloring, respectively, because of the self-duality.

Using the same number of colors as the number of vertices, the number of vertex-coloring, obtained with the above indices, is shown in Table 2. Likewise, the number of edge-, face-, and cell-colorings are shown in Table 2 as well.

Table 2. Number of coloring for regular polytope

5-cell	
vertex, cell	127
edge, face	166920040
8-cell (16-cell)	
vertex (cell)	96076862179356736
edge (face)	7611987694431786032465720642117699749854576640
face (edge)	6946540504430644998370054972800
cell (vertex)	102726
24-cell	
vertex, cell	2315513501476956735839749356000
edge, face	34483861201359023577389107012981965928313060 57838791880900258968549927553141895551501201 46025625062983503708752006856211231161205525 08537727234555358979650823158751389463486852 267157487616

For the regular polytope of 600-cell, we could not obtain the rotation group G by using computer, because the ratio of |G|=7200 and the number of permutations 120! of its 120 vertices is very small. Since we create the rotation group G for the polytope by permutation of its vertices by using computer, we can enumerate any kind of polytope if the number of its vertices is not as large as that of the 600-cell.

REFERENCES

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Riordan, J. (1978) *An Introduction to Combinatorial Analysis*, Princeton University Press, Princeton, New Jersey.